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Hunting for the central charge of the Virasoro algebra: six- and eight-state spin models

G von Gehlen and V Rittenberg
Physikalisches Institut, Universität Bonn, Nussallee 12, D-5300 Bonn 1, West Germany

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Abstract. In two dimensions conformal invariance has important implications for the finite-size scaling properties of the spectra of transfer matrices and quantum chains at the critical point. Some relations between the finite-size scaling amplitudes are obtained which can be used as a test of conformal invariance and implicitly to distinguish between second-order and first-order phase transitions. The numerical values of the scaling amplitudes give the scale dimensions of various operators.

Six- and eight-state self-dual quantum chains with cubic symmetries are considered at the critical point for three values of the coupling constant. The systems are found to be conformal invariant and estimates for several critical exponents are obtained. Based on the approximate values of the critical exponent one tries to find the values of the corresponding central charges of the Virasoro algebras.

1. Introduction

It was shown by Cardy (1984a) using conformal invariance that for a strip with N lattice spacings and *periodic boundary conditions* the inverse correlation length κ behaves at the critical point of the infinite system like

$$\kappa = AN^{-1} \tag{1.1}$$

with the finite-size scaling amplitude A

$$A = 2\pi x \tag{1.2}$$

where x is the scale dimension of the operator concerned. In this way an earlier observation by Luck (1982), Derrida and de Seze (1982), Nightingale and Blöte (1983) and Privman and Fisher (1984) is explained. It was subsequently shown by Cardy (1984b) that for other boundary conditions the proportionality factor in (1.1) is different but is still related to the critical exponents like the surface exponents (free boundary conditions) or the scale dimensions of 'para-fermionic' operators† ('antiperiodic' boundary conditions).

Subsequently conformal invariance was applied to the finite-size scaling properties of quantum chains (Burkhart and Guim 1985, von Gehlen *et al* 1986) and the corresponding finite-size scaling amplitudes have been determined.

An important consequence of this study is that for the energy–energy correlations with free and 'antiperiodic' boundary conditions (in general there are several of them) the amplitudes A in (1.1) are all equal to 2π providing a test of conformal invariance.

† Para-fermions in statistical mechanics (Fradkin and Kadanoff 1980) and particle physics (Bradcken and Green 1972) are not obviously related objects.

An immediate application of this observation is that we have in our hand a method which allows us to decide from finite-size scaling if a transition is first or second order. Indeed, if in numerical calculations we find the amplitudes close to 2π, we have a good reason to believe that the transition is second order. A second method to check conformal invariance is obviously (when considering other correlation functions) to compare the numerical values of A with $2\pi x$ where x is known from other sources.

In this paper we use conformal invariance combined with finite-size scaling in order to settle the controversial issue of the nature of the phase transition for the six- and eight-state spin systems with cubic symmetry (Kim *et al* 1975, Aharony 1977). Using the vacancy generating renormalisation group transformation, Nienhuis *et al* (1983) have found that the phase transition is first order. A similar result was obtained by Igloi (1986a, b). On the other hand Monte Carlo calculations by Badke *et al* (1985) suggest a second-order phase transition. This result was supported by Patkos and Rujan (1985). In the present paper we consider self-dual quantum chains with cubic symmetry and do a finite-size scaling analysis using up to eight sites for the six-state model and seven sites for the eight-state model. In both cases we find that we have conformal invariance and thus the transitions are second order. At the same time several estimates for the critical exponents are obtained.

At this point one can push the understanding of the problem one step further and obtain the *exact* values of the critical exponents. In principle this can be done if we guess the value of the central charge of the Virasoro algebra (Belavin *et al* 1984), assume unitary representations (Friedan *et al* 1984) and assign the various operators which appear in finite-size scaling to certain irreducible representations of the Virasoro algebra (Dotsenko 1984, Cardy 1984c, von Gehlen *et al* 1986). This procedure might work if the estimates for the critical exponents are much more precise than ours, since for a given central charge there are many possible anomalous dimensions and their values vary smoothly with the central charge. We have thus adopted a different philosophy based on our experience with the Potts (1952) model (von Gehlen *et al* 1985). We have guessed the representations of the Virasoro algebra corresponding to some operators and have *tried* to determine the central charge from the numerical values of the finite-size scaling amplitudes. As will be seen our guess was right and we have obtained a consistent picture of the anomalous dimensions but their values are rather insensitive to the precise value of the central charge. Similar problems are bound to happen with other spin systems unless one finds on conceptual grounds a connection between the properties of the spin system and the value of the central charge of the Virasoro algebra.

The paper is organised as follows. In § 2 we describe the model and its symmetry properties. In § 3 we summarise the connection between the finite-size scaling amplitudes of quantum chains and the critical exponents. Section 4 describes briefly our assumptions about the correspondence between various operators and the irreducible representations. In § 5 we present the estimates for the finite-size scaling amplitudes, we check the conformal invariance of our systems and make the connection between our estimates for the critical exponents and the Virasoro algebra. Our conclusions are presented in § 6.

2. The cubic symmetric self-dual Hamiltonian

Recently Badke *et al* (1985) have studied the two-dimensional six- and eight-state spin models with cubic symmetry using the Monte Carlo method and have claimed to observe second-order phase transitions with

$$x_t \approx \tfrac{1}{2} \qquad x_h \approx 0.15 \qquad \text{(six-state model)} \tag{2.1}$$

$$x_t \approx \tfrac{1}{3} \qquad x_h \approx 0.15 \qquad \text{(eight-state model).} \tag{2.2}$$

Using the Hamiltonian version of the model and finite-size scaling Patkos and Rujan (1985) also obtain $x_t \sim \frac{1}{2}$ for the six-state model. This contrasts with the results of Nienhuis *et al* (1983) and Igloi (1986a, b) who claim first-order phase transitions.

In order to settle the controversy we consider here the self-dual Hamiltonian version of the model. In this way the critical point is known and the determination of the finite-size scaling amplitudes is thus more precise.

The N-site Hamiltonian with n states, '$\tilde{Q}$' boundary condition (von Gehlen *et al* 1985) and cubic symmetry is

$$\begin{aligned}H^{(\tilde{Q})} = &-\alpha \sum_{k=0}^{t-1} \sum_{i=1}^{N} (\sigma_i)^{2k+1} - \beta \sum_{k=0}^{t-2} \sum_{i=1}^{N} (\sigma_i)^{2k+2} \\ &- \gamma \sum_{k=0}^{t-1} \sum_{i=1}^{N-1} (\Gamma_i)^{2k+1} (\Gamma_{i+1})^{n-2k-1} - \delta \sum_{k=0}^{t-2} \sum_{i=1}^{N-1} (\Gamma_i)^{2k+2} (\Gamma_{i+1})^{n-2k-2} \\ &- \gamma \sum_{k=0}^{t-1} (\Gamma_N)^{2k+1} (\omega^{\tilde{Q}} \Gamma_1)^{n-2k-1} - \delta \sum_{k=0}^{t-2} (\Gamma_N)^{2k+2} (\omega^{\tilde{Q}} \Gamma_1)^{n-2k-2}\end{aligned} \tag{2.3}$$

where

$$n = 2t \qquad \omega = \exp(2\pi \mathrm{i}/n) \tag{2.4}$$

$$\sigma_i = 1 \otimes 1 \otimes \ldots \otimes \underset{\text{`}i\text{th' site}}{\sigma} \otimes 1 \ldots \otimes 1 \tag{2.5a}$$

$$\Gamma_i = 1 \otimes 1 \otimes \ldots \otimes \Gamma \otimes 1 \ldots \otimes 1 \tag{2.5b}$$

$$\sigma = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 0 & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \tag{2.6}$$

$$\sigma^n = \Gamma^n = 1. \tag{2.7}$$

For $\tilde{Q} = 0$ we have periodic boundary conditions and for $\tilde{Q} = 1, \ldots, n-1$ we have 'antiperiodic' boundary conditions. The Hamiltonian corresponding to the free boundary condition ($H^{(\mathrm{F})}$) is obtained by dropping the last two terms in (2.3).

In (2.3) α, β, γ and δ are coupling constants and the symmetry of the Hamiltonians (2.3) is $Z_2 \sim S_t$ (the wreath product of Z_2 and the permutation group of t objects S_t) which is a group of order $2^t t!$ (Badke *et al* 1985).

The Hamiltonians $H^{(\tilde{Q})}$ and $H^{(\mathrm{F})}$ commute with the charge operator $\hat{Q}$

$$\hat{Q} = \sum_{i=1}^{N} q_i (\mathrm{mod}\ n) \tag{2.8}$$

where

$$q = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & n-1 \end{pmatrix} \tag{2.9}$$

and we denote the eigenvalues of $\hat{Q}$ by $Q=0,1,\ldots,n-1$. Because of the charge conservation, the Hamiltonians $H^{(\tilde{Q})}$ and $H^{(\mathrm{F})}$ split into n charge sectors and we will denote the corresponding matrices by $H_Q^{(\tilde{Q})}$ and $H_Q^{(\mathrm{F})}$.

The Hamiltonians $H^{(\tilde{Q})}$ are self-dual if

$$\beta=\alpha\epsilon \qquad \gamma=\alpha\lambda \qquad \delta=\alpha\varepsilon\lambda. \tag{2.10}$$

For convenience we choose the normalisation factor α in (2.10) to be

$$\alpha=2/n(1+\varepsilon) \tag{2.11}$$

and we are left with two coupling constants λ and ε. The n-state Potts model (Potts 1952) is obtained for $\varepsilon=1$.

The four-state model was studied by Kohmoto *et al* (1981) and Igloi and Sólyom (1984). They have shown that for $-\sqrt{2}/2\leqslant\varepsilon\leqslant 1$ there is a single phase transition at $\lambda=1$ with a fixed critical exponent $x_h=\frac{1}{8}$ and a 'running' critical exponent

$$x_t=\tfrac{1}{2}\left(1-\frac{\cos^{-1}\varepsilon}{\pi}\right)^{-1}. \tag{2.12}$$

In the present paper we consider the cases $n=6$ and 8. We will assume the critical point to be at $\lambda=1$ and will take three values for ε ($\varepsilon=0$, $\frac{1}{3}$ and $\frac{3}{5}$). We will perform a complete finite-size scaling study of the systems as explained in § 3. Such a study has not yet been done in the $n=4$ case. Before proceeding with our finite-size scaling analysis let us mention some symmetry properties which follow from the cubic symmetry of the Hamiltonians and self-duality at $\lambda=1$.

(a) $n=6$

$$H_Q^{(\tilde{Q})}=H_{\tilde{Q}}^{(Q)} \qquad H_2^{(\tilde{Q})}=H_4^{(\tilde{Q})} \qquad H_1^{(\tilde{Q})}=H_5^{(\tilde{Q})} \tag{2.13a}$$

$$H_1^{(\mathrm{F})}=H_5^{(\mathrm{F})} \qquad H_2^{(\mathrm{F})}=H_4^{(\mathrm{F})} \tag{2.13b}$$

(there are ten independent matrices $H_Q^{(\tilde{Q})}$ and four independent matrices $H_Q^{(\mathrm{F})}$).

(b) $n=8$

$$\begin{gathered} H_Q^{(\tilde{Q})}=H_{\tilde{Q}}^{(Q)} \qquad H_1^{(\tilde{Q})}=H_7^{(\tilde{Q})} \qquad H_2^{(\tilde{Q})}=H_6^{(\tilde{Q})} \qquad H_3^{(\tilde{Q})}=H_5^{(\tilde{Q})} \\ H_1^{(2k)}=H_3^{(2k)}=H_5^{(2k)}=H_7^{(2k)} \qquad k=0,1,2,3 \end{gathered} \tag{2.14a}$$

$$H_1^{(\mathrm{F})}=H_3^{(\mathrm{F})}=H_5^{(\mathrm{F})}=H_7^{(\mathrm{F})} \qquad H_2^{(\mathrm{F})}=H_6^{(\mathrm{F})} \tag{2.14b}$$

(there are eleven independent matrices $H_Q^{(\tilde{Q})}$ and four independent matrices $H_Q^{(\mathrm{F})}$).

We are in a position to see the implications of conformal invariance on the spectra of the matrices $H_Q^{(\tilde{Q})}$ and $H_Q^{(\mathrm{F})}$.

3. Conformal invariance and finite one-dimensional quantum chains

In this section we summarise the results obtained by von Gehlen *et al* (1985) on the finite-size scaling properties of quantum chains at the critical point.

Let

$$x_A=\Delta_A+\bar{\Delta}_A \qquad s_A=\bar{\Delta}_A-\Delta_A \tag{3.1}$$

be the scale dimensions (x_A) and spin (s_A) of an operator ϕ_A. The two-point correlation function is

$$\langle\phi_A(z_1,\bar{z}_1)\phi_{A'}(z_2,\bar{z}_2)\rangle=\delta_{AA'}(z_1-z_2)^{-2\Delta_A}(\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}_A}. \tag{3.2}$$

Here

$$z = x + \mathrm{i}y \qquad \bar{z} = x - \mathrm{i}y \tag{3.3}$$

x, y are the coordinates in the plane.

We now define the notation. We start with spinless operators. We denote by x_t the scale dimensions of the energy density, by $x_Q (Q = 1, \ldots, n-1)$ the scale dimensions of the order operators σ_Q and by $x_{Q,s}$ the surface exponents (Binder 1983) of the order operators. Since our systems are self-dual, the scale dimensions of the order operators σ_Q and disorder operators $\mu_{\tilde{Q}}$ are the same.

The short-distance product of the operators σ_Q and $\mu_{\tilde{Q}}$ generates the 'parafermionic' operator $\psi_{Q,\tilde{Q}}$ (Fradkin and Kadanoff 1980) with scale dimensions $x_{Q,\tilde{Q}}$ and spin $S_{Q,\tilde{Q}}$.

We denote by $E_Q^{(\tilde{Q})}(r)$, $r = 0, 1, 2, \ldots$, the eigenvalues of the matrices $H_Q^{(\tilde{Q})}$ defined in the previous section. $E_Q^{(\tilde{Q})}(0)$ corresponds to the ground state, $E_Q^{(\tilde{Q})}(1)$ to the first excited state, etc. Similarly $E_Q^{(\mathrm{F})}(r)$, $r = 0, 1, 2, \ldots$, represent the eigenvalues of the matrices $H_Q^{(\mathrm{F})}$. The eigenvalues $E_Q^{(\tilde{Q})}(r)$ and $E_Q^{(\mathrm{F})}(r)$ correspond to the N-site problem and their values depend on N. As shown by von Gehlen *et al* (1985) following Cardy (1984a, b, c) we have the following results for the finite-size scaling amplitudes R (corresponding to the energy-energy correlations) and P (corresponding to the spin-spin correlations):

$$R^{(0)} = \lim_{N\to\infty} N(E_0^{(0)}(1) - E_0^{(0)}(0)) = 2\pi\xi x_t \tag{3.4a}$$

$$R^{(\tilde{Q})} = \lim_{N\to\infty} N(E_0^{(\tilde{Q})}(1) - E_0^{(\tilde{Q})}(0)) = 2\pi\xi \qquad (\tilde{Q} = 1, \ldots, n-1) \tag{3.4b}$$

$$R^{(\mathrm{F})} = \lim_{N\to\infty} N(E_0^{(\mathrm{F})}(1) - E_0^{(\mathrm{F})}(0)) = 2\pi\xi \tag{3.4c}$$

$$P_0^{(Q)} = \lim_{N\to\infty} N(E_Q^{(0)}(0) - E_0^{(0)}(0)) = 2\pi\xi x_Q \qquad (Q = 1, \ldots, n-1) \tag{3.5a}$$

$$P_Q^{(\mathrm{F})} = \lim_{N\to\infty} N(E_Q^{(\mathrm{F})}(0) - E_0^{(\mathrm{F})}(0)) = 2\pi\xi x_{Q,s}/2 \qquad (Q = 1, \ldots, n-1) \tag{3.5b}$$

$$P_Q^{(\tilde{Q})} = \lim_{N\to\infty} N(E_Q^{(\tilde{Q})}(0) - E_0^{(0)}(0)) = 2\pi\xi x_{Q,\tilde{Q}} \qquad (\tilde{Q}, Q \neq 0). \tag{3.6}$$

The constant ξ in (3.4) and (3.5) appears because the normalisation of the quantum chains is free. (The constant α in (2.3) can be chosen at will; see (2.11).)

The information provided by (3.4)–(3.6) can be divided into two parts. The equality of the scaling amplitudes $R^{(\tilde{Q})}$ ($\tilde{Q} \neq 0$) and $R^{(\mathrm{F})}$ can be used as a check of conformal invariance. If they are indeed equal, one can use their common value to determine ξ and in this way to obtain x_t (3.4a), x_Q (3.5a), $x_{Q,s}$ (3.5b) and $x_{Q,\tilde{Q}}$ (3.6). Notice that the finite-size scaling amplitudes provide us with information only on the scale dimensions of the operators and not on their spin. In the next section we will see that one can relate the spin and the scale dimensions.

We will postpone the finite-size scaling study of the six- and eight-state systems until § 5. In the next section we will give an outlook on which kinds of scale dimensions we can expect. This discussion is based on our experience with the two-, three- and four-state Potts models (von Gehlen *et al* 1985).

4. Anomalous dimensions and conformal invariance

Let us assume that we have found that our spin systems are conformal invariant and that they correspond to a certain central charge $c \leqslant 1$ of the Virasoro algebra (Belavin *et al* 1984, Friedan *et al* 1984). If we require also unitarity, the anomalous dimensions Δ and $\bar{\Delta}$ (see (3.2)) are rational numbers $\Delta_{p,q}$ and $\bar{\Delta}_{p',q'}$ where

$$\Delta_{p,q}=\frac{[(m+1)p-mq]^2-1}{4m(m+1)} \qquad \bar{\Delta}_{p',q'}=\frac{[(m+1)p'-mq']^2-1}{4m(m+1)} \tag{4.1}$$

$$(1 \leqslant p,\ p' \leqslant m-1,\ 1 \leqslant q,\ q' \leqslant m)$$

$$c=1-\frac{6}{m(m+1)} \qquad m=3,4,\ldots . \tag{4.2}$$

(Notice that these considerations do not apply to the four-state model of Kohmoto *et al* (1981) since in this case $c=1$ as in the Thirring model (Fubini *et al* 1973) and x_t varies *continuously* with ε (see (2.12).)

Our next task is to assign to the various operators considered in § 3 the corresponding values $\Delta_{p,q}$ and $\bar{\Delta}_{p',q'}$. We again follow von Gehlen *et al* (1985) and study separately the cases m odd and even.

(a) *m odd.* We assume (Dotsenko 1984) that all order parameters σ_Q have the same anomalous dimensions

$$x_Q=x_h=2\Delta_{(m+1)/2,(m+1)/2}=\frac{(m+3)(m-1)}{8m(m+1)} \tag{4.3}$$

and

$$x_t=2\Delta_{2,1}=(m+3)/2m. \tag{4.4}$$

For the surface exponent we take (Cardy 1984c)

$$\frac{x_{Q,s}}{2}=\frac{x_{h,s}}{2}=\frac{\Delta_{1,3}}{2}=\frac{1}{2}\frac{(m-1)}{(m+1)}. \tag{4.5}$$

The two-, three- and four-state Potts models correspond to $m=3$, 5 and ∞.

We assume, following Fradkin and Kadanoff (1980), that for a system defined on a Z_p cyclic group (our six- and eight-state models are defined on Z_6 (respectively Z_8) (see (2.7))), the 'para-fermionic' operators $\psi_{Q,\tilde{Q}}$ have a spin

$$s_{Q,\tilde{Q}}=Q\tilde{Q}/p. \tag{4.6}$$

This assumption was checked in the three-state Potts model where $s_{1,1}$ was found to be $\frac{1}{3}$ and in the four-state Potts model where $s_{1,1}$ and $s_{1,2}$ were found to be $\frac{1}{4}$ (respectively $\frac{2}{4}$). The Fradkin–Kadanoff conjecture failed however for $s_{2,2}$ where one expects $s_{2,2}=1$ and it was found that

$$x_{2,2}>s_{2,2} \qquad x_{2,2}\simeq 0.45. \tag{4.7}$$

From now on we will assume the relation (4.6) to be valid for $s<1$.

It was noticed by von Gehlen *et al* (1985) that for

$$p=(m+1)/2 \tag{4.8}$$

one has

$$s = l/p = \bar{\Delta}_{2l,2l-1} - \Delta_{m-2l,m-2l} \tag{4.9a}$$

$$x(s) = \bar{\Delta}_{2l,2l-1} + \Delta_{m-2l,m-2l} \tag{4.9b}$$

$$= \tfrac{1}{2} + [s^2(m+1) - 1]/2m \qquad l = 1, 2, \ldots, p-1.$$

From (4.9*b*) we obtain the bounds

$$1 \geqslant x(s) \geqslant \tfrac{1}{3}. \tag{4.10}$$

We would like to stress that the choice (4.9) is not unique. Let us consider the case $m = 11$ as an example and look for operators with spin $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$ and $\frac{4}{6}$. Using (4.1) we find that there is only one operator having spin $\frac{1}{6}$ and another one having spin $\frac{2}{6}$ but there are two operators having spin $\frac{3}{6}$ and two for $\frac{4}{6}$. For spin $\frac{3}{6}$ we have one operator given by (4.9) and another one with anomalous dimensions

$$\bar{\Delta}_{3,1} - \Delta_{8,7} = \tfrac{3}{6} \qquad \bar{\Delta}_{3,1} + \Delta_{8,7} \simeq 1.86. \tag{4.11}$$

For spin $\frac{4}{6}$ we have one operator given by (4.9) and another operator

$$\bar{\Delta}_{7,6} - \Delta_{6,4} = \tfrac{4}{6} \qquad \bar{\Delta}_{7,6} + \Delta_{6,4} \simeq 1.89. \tag{4.12}$$

The finite-size scaling amplitudes will show us whether our assumption (equation (4.9)) was right.

(b) *m even.* Here we take

$$x_Q = x_h = 2\Delta_{m/2,m/2} = \frac{m^2-4}{8m(m+1)} \tag{4.13}$$

$$x_t = 2\Delta_{1,2} = \frac{m-2}{2(m+1)} \tag{4.14}$$

$$\frac{x_{Q,s}}{2} = \frac{x_{h,s}}{2} = \frac{1}{2}\Delta_{3,1} = \frac{m+2}{2m} \tag{4.15}$$

and for the 'para-fermionic' operators

$$p = \tfrac{1}{2}m \qquad s = l/p \qquad l = 1, 2, \ldots, p-1 \tag{4.16}$$

$$s = \bar{\Delta}_{2l+1,2l} - \Delta_{m-2l+1,m-2l+1} \tag{4.17a}$$

$$x(s) = \bar{\Delta}_{2l+1,2l} + \Delta_{m-2l+1,m-2l+1} \tag{4.17b}$$

$$= \frac{1}{2} + \frac{s^2 m + 1}{2(m+1)}.$$

From (4.17*b*) we obtain the bounds

$$1 \geqslant x(s) \geqslant \tfrac{1}{2}. \tag{4.18}$$

The tricritical n-state Potts models correspond to $m = 4$ $(n = 2)$, $m = 6$ $(n = 3)$ and $m = \infty$ $(n = 4)$.

Since in the next section we are going to 'hunt' for the central charge c of the Virasoro algebra from the knowledge of the scale dimensions, in figure 1 we display x_t, x_h and $\frac{1}{2}x_{h,s}$ as a function of m. We notice that $x_h \simeq 0.1$ for all the values of m. The exponents x_y and $x_{h,s}$ have a stronger dependence on m but again for $m > 15$ the dependence is weak. In view of the application to the six-state model ($p = 6$), in figure 2 we display $x(s)$ for $s = \frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$ and $\frac{4}{6}$. We notice again a weak dependence on m but one qualitative picture emerges: the values of $x(s)$ increase with s and they all cluster around $x(s) \simeq 0.6$. Finally in figure 3 we give $x(s)$ for $s = \frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$ and $\frac{4}{8}$. These values

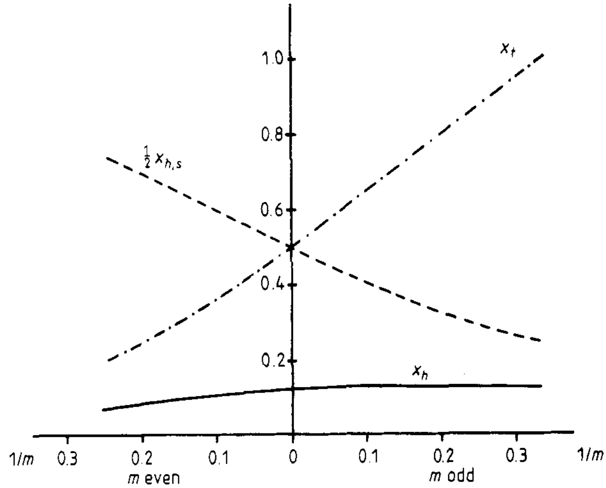

Figure 1. The scale dimensions x_t, x_h and $\frac{1}{2}x_{h,s}$ as a function of m according to (4.3) and (4.13)–(4.15).

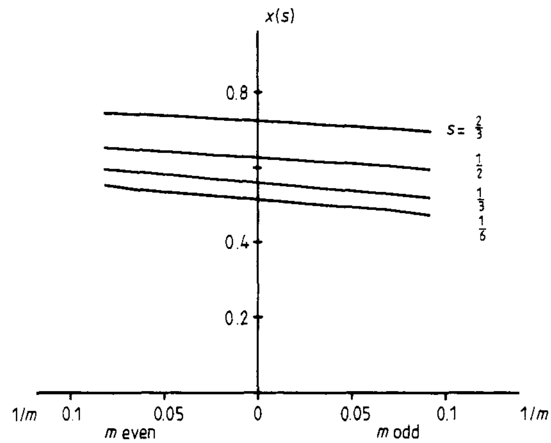

Figure 2. The scale dimensions $x(s)$ for 'para-fermions' of spin s ((4.9) and (4.17)) as a function of m. The figure is relevant for the six-state model.

will be useful for the eight-state model. The qualitative picture is the same as in figure 2.

In §§ 3 and 4 we have gathered together the necessary ingredients for the analysis of the six- and eight-state systems to which we now proceed.

5. Numerical results for the six- and eight-state cubic models

We now present the numerical results for the finite-size scaling amplitudes. As mentioned earlier, we have considered only the self-dual case (see equation (2.10)) of the Hamiltonian (2.3) and because of the large amount of computer work involved we have taken only three values for the coupling constant ε: 0, $\frac{1}{3}$ and $\frac{3}{5}$. All calculations were done for $\lambda = 1$.

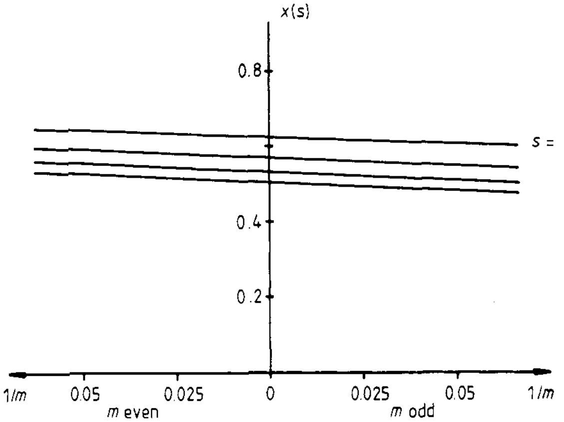

Figure 3. The scale dimensions $x(s)$ for 'para-fermions' of spin s as a function of m. The figure is relevant for the eight-state model.

The independent finite-size amplitudes (see equations (2.13) and (2.14)) are presented in tables 1-4 only for two values of ε. N indicates the number of sites of the quantum chains and we have used the definitions given by (3.4)-(3.6). From these tables we have obtained the asymptotic (large N) values of the finite-size scaling amplitudes using the Van den Broeck-Schwartz (1979) approximants and power fits. The estimates for the finite-size scaling amplitudes for all three values of ε are given in tables 5 and 6. A dash corresponds to amplitudes which have not been computed.

We first check for conformal invariance. According to (3.4*b*) and (3.4*c*) we should have

$$R^{(2)} = R^{(3)} = R^{(F)} \tag{5.1}$$

for the six-state model and

$$R^{(2)} = R^{(4)} = R^{(F)} \tag{5.2}$$

for the eight-state model. From tables 5 and 6 we see that the relations (5.1) and (5.2) are satisfied for the three values of ε and thus *the systems are conformal invariant and the phase transitions are second order.* This conclusion should be taken as usual with a caveat. One can never rule out the possibility of a weak first-order phase transition with a correlation length much larger than the size of our chains such that the approximate validity of the relations (5.1) and (5.2) is an artefact of the small values of N.

We estimate the value of $2\pi\xi$ (see equations (3.4*a*, *b*)) to be 4.9 ($\varepsilon = 0$), 4.8 ($\varepsilon = \frac{1}{3}$) and 4.7 ($\varepsilon = \frac{3}{5}$) for the six-states model and 4.8 ($\varepsilon = 0$), 4.7 ($\varepsilon = \frac{1}{3}$) and 4.65 ($\varepsilon = \frac{3}{5}$) for the eight-state model. We now use (3.4*a*), (3.5*a*, *b*) and (3.6) to determine the scale dimensions. In table 7 we give the normalised values of the finite-size scaling amplitudes for the six-state model

$$\bar{R}^{(0)} = R^{(0)}/2\pi\xi \qquad \bar{R}^{(2)} = R^{(2)}/2\pi\xi \ldots . \tag{5.3}$$

The normalised values are equal to the scale dimensions (the errors in the table are very subjective). We now discuss the critical exponents. First the order operators: we note that if $x_1 = \bar{P}_0^{(1)}$ is constant (independent of ε), for the other two $x_2 = \bar{P}_0^{(2)}$ and $x_3 = \bar{P}_0^{(3)}$, we have

$$x_2 \simeq x_3 \neq x_1 \tag{5.4}$$

Table 1. Finite-size scaling amplitudes ((3.4)–(3.6)) for the six-state model with $\varepsilon = \frac{3}{5}$. N indicates the number of sites of the quantum chain.

(a)

N	$R^{(0)}$	$R^{(2)}$	$R^{(3)}$	$R^{(F)}$
2	3.342 703 268	2.782 951 712	3.356 682 202	3.120 971 442
3	3.097 699 012	3.610 198 613	4.219 913 101	3.562 277 358
4	2.912 346 252	3.956 038 839	4.526 642 045	3.796 592 588
5	2.774 659 182	4.137 524 193	4.658 174 850	3.940 922 004
6	2.667 678 039	4.247 657 090	4.722 399 645	4.038 877 440
7	2.581 146 340	4.321 405 446	4.756 525 144	
8	2.508 914 826	4.374 390 736	4.775 723 157	

(b)

N	$P_1^{(1)}$	$P_2^{(1)}$	$P_3^{(1)}$	$P_2^{(2)}$	$P_3^{(2)}$	$P_3^{(3)}$
2	1.956 643 771	2.200 567 413	2.771 652 582	2.256 217 806	2.337 844 097	3.157 959 355
3	1.990 216 813	2.302 663 569	2.899 790 647	2.410 079 121	2.451 943 000	3.316 860 207
4	2.017 281 208	2.363 998 332	2.951 050 395	2.508 177 023	2.528 219 212	3.377 466 549
5	2.039 271 510	2.407 077 399	2.978 603 541	2.578 184 089	2.582 833 764	3.407 477 214
6	2.058 016 053	2.440 196 919	2.996 520 749	2.631 750 481	2.624 475 582	3.425 108 906
7	2.074 576 010	2.467 154 938	3.009 765 863	2.674 687 019	2.657 735 409	3.436 839 168
8	2.089 573 747	2.489 964 907	3.020 444 420	2.710 264 410	2.685 232 896	3.445 417 189

(c)

N	$P_1^{(0)}$	$P_2^{(0)}$	$P_3^{(0)}$	$P_1^{(F)}$	$P_2^{(F)}$	$P_3^{(F)}$
2	0.703 745 9973	0.741 033 0467	0.886 526 1499	1.349 007 104	1.405 882 093	1.677 676 838
3	0.645 583 9598	0.696 325 9431	0.807 667 8415	1.534 931 509	1.634 941 537	1.902 551 047
4	0.618 946 3983	0.676 530 3135	0.768 918 6681	1.656 810 490	1.788 812 381	2.046 755 361
5	0.602 787 6648	0.664 195 7135	0.744 155 5901	1.746 182 999	1.902 643 698	2.150 225 711
6	0.591 499 5498	0.655 101 2024	0.726 153 1891	1.816 396 561	1.992 114 077	2.229 901 180
7	0.582 921 7148	0.647 748 0023	0.712 040 6037			
8	0.576 031 9296	0.641 469 9651	0.700 422 9544			

for both $\varepsilon = 0$ and $\varepsilon = \frac{1}{3}$ and $x_2 \simeq x_3$ varies with ε. Thus similar to the four-state model (see equation (2.12)) *the scale dimensions depend on the coupling constant* ε. The same observation is valid for $x_t = \bar{R}(0)$, $x_{1,1} = \bar{P}_1(1)$, etc.

In order to obtain more confidence in our estimates and to have an independent check of conformal invariance we have computed x_t from the energy gap $G(\lambda, N)$ of the N-site Hamiltonian with periodic boundary conditions. This calculation was done for the six-state problem with $\varepsilon = 0$. Using (Barber 1983)

$$x_t = 2 - \ln\left(\frac{\mathrm{d}G(\lambda, N+1)}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}G(\lambda, N)}{\mathrm{d}\lambda}\right)^{-1}\right)_{\lambda=1} [\ln((N+1)/N)]^{-1} \tag{5.5}$$

and chains with up to eight sites, we have obtained

$$x_t \simeq 0.67 \tag{5.6}$$

in excellent agreement with the value quoted in table 7.

Table 2. Finite-size scaling amplitudes for the six-state model with $\varepsilon = 0$.

(*a*)

N	$R^{(0)}$	$R^{(2)}$	$R^{(3)}$	$R^{(\mathrm{F})}$
2	3.265 986 324	2.431 465 694	4.000 000 000	3.055 050 463
3	3.217 130 713	3.273 486 086	4.225 450 834	3.574 321 834
4	3.166 560 431	3.719 487 494	4.525 136 958	3.859 176 407
5	3.129 092 960	3.988 802 304	4.651 846 733	4.034 680 822
6	3.102 392 079	4.163 139 479	4.711 019 987	4.151 974 834
7	3.083 454 761	4.282 319 417	4.739 530 329	4.235 144 755
8	3.070 079 156	4.366 862 294	4.752 628 479	4.296 805 440

(*b*)

N	$P_1^{(1)}$	$P_2^{(1)}$	$P_3^{(1)}$	$P_2^{(2)}$	$P_3^{(2)}$	$P_3^{(3)}$
2	1.732 181 543	2.168 161 307	3.284 476 879	2.456 049 754	3.284 476 879	5.284 476 879
3	1.652 725 453	2.252 364 865	3.274 938 517	2.817 735 833	3.470 028 577	5.116 110 198
4	1.608 310 366	2.296 660 485	3.252 158 368	3.059 151 377	3.587 297 107	5.018 872 819
5	1.578 458 502	2.322 272 148	3.233 204 674	3.228 135 970	3.669 874 272	4.957 896 917
6	1.556 292 009	2.337 902 352	3.218 787 271	3.352 276 863	3.732 373 347	4.920 138 710
7	1.538 743 609	2.347 716 992	3.207 871 512	3.447 211 037	3.782 092 620	4.897 046 765
8	1.524 221 156	2.353 926 488	3.199 500 969	3.522 223 548	3.823 101 279	4.883 437 575

(*c*)

N	$P_1^{(0)}$	$P_2^{(0)}$	$P_3^{(0)}$	$P_1^{(\mathrm{F})}$	$P_2^{(\mathrm{F})}$	$P_3^{(\mathrm{F})}$
2	0.668 247 5505	0.853 011 1846	1.284 476 879	1.266 661 976	1.508 706 152	2.296 399 853
3	0.597 495 3545	0.847 863 6068	1.146 096 718	1.383 360 885	1.808 689 853	2.521 452 175
4	0.567 288 7580	0.852 339 9812	1.092 943 573	1.447 513 690	2.004 689 443	2.658 115 409
5	0.550 901 4075	0.857 801 0556	1.067 927 361	1.487 226 533	2.142 075 856	2.751 411 146
6	0.540 705 0007	0.863 023 0125	1.055 173 117	1.513 587 760	2.243 366 560	2.820 177 580
7	0.533 752 6146	0.867 839 2397	1.048 647 834	1.531 861 182	2.320 924 235	2.873 567 770
8	0.528 690 8237	0.872 276 2130	1.045 617 017	1.544 870 816	2.382 073 508	2.916 584 149
9	0.524 820 0942	0.876 390 7887	1.044 687 159			

There is finally another argument for the case of a second-order transition. The estimates for the scale dimension x_1 of the order parameter have small errors and they give $x_1 \simeq 0.11$ which is a typical figure for this operator (see the Kohmoto *et al* (1981) model mentioned in § 2 and figure 1). For a first-order transition one would expect $x_1 = 0$ (Fisher and Berker 1982, Privman and Fisher 1983).

We now turn to the problem of the central charge. We notice that for $\varepsilon = \frac{3}{5}$ we can take

$$x_1 \approx x_2 \approx x_3 \approx x_h \qquad (5.7)$$
$$\tfrac{1}{2}x_{1,s} = \bar{P}_1^{(\mathrm{F})} \approx \tfrac{1}{2}x_{2,s} = \bar{P}_2^{(\mathrm{F})} \approx \tfrac{1}{2}x_{3,s} = \bar{P}_3^{(\mathrm{F})} = \tfrac{1}{2}x_{h,s}.$$

We also have (see equations (4.6), (4.9) and (4.17))

$$x(\tfrac{1}{6}) = \bar{P}_1^{(1)} \qquad x(\tfrac{2}{6}) = \bar{P}_1^{2} \qquad x(\tfrac{3}{6}) = \bar{P}_1^{(3)} \qquad x(\tfrac{4}{6}) = \bar{P}_2^{(2)} \qquad (5.8)$$

where $x(s)$ is the scale dimension of an operator of spin s. An inspection of figures 1 and 2 teaches us that one obtains a consistent picture for the exponents taking m

Table 3. Finite-size scaling amplitudes for the eight-state model with $\varepsilon = \frac{3}{5}$.

(*a*)

N	$R^{(0)}$	$R^{(2)}$	$R^{(4)}$	$R^{(F)}$
2	2.916 624 852	2.641 218 744	2.990 403 673	2.894 977 435
3	2.569 510 466	3.420 115 659	3.815 152 049	3.307 774 952
4	2.329 492 261	3.748 885 126	4.128 706 796	3.532 401 445
5	2.153 782 144	3.924 267 827	4.278 742 414	3.675 635 026
6	2.016 741 405	4.033 499 918	4.363 361 820	3.777 046 548
7	1.904 791 706	4.109 323 824	4.417 379 088	

(*b*)

N	$P_1^{(1)}$	$P_2^{(1)}$	$P_3^{(1)}$	$P_4^{(1)}$	$P_2^{(2)}$	$P_4^{(2)}$	$P_4^{(4)}$
2	1.957 853 002	2.097 273 916	2.209 900 471	2.709 310 025	2.407 405 310	2.381 771 373	2.778 849 175
3	2.031 930 246	2.205 159 526	2.315 758 792	2.862 903 513	2.586 785 352	2.509 486 429	2.935 829 645
4	2.097 364 287	2.283 629 896	2.385 369 071	2.939 027 996	2.691 068 745	2.590 236 010	3.012 661 196
5	2.153 853 141	2.345 655 713	2.439 707 457	2.988 050 346	2.763 995 848	2.647 160 367	3.056 473 301
6	2.203 829 391	2.397 736 459	2.485 814 127	3.024 934 530	2.820 611 987	2.691 088 096	3.082 890 445
7	2.249 048 287	2.443 275 295	2.526 771 902	3.055 452 550	2.867 516 099	2.727 295 248	3.102 288 772

(*c*)

N	$P_1^{(0)}$	$P_2^{(0)}$	$P_4^{(0)}$	$P_1^{(F)}$	$P_2^{(F)}$	$P_4^{(F)}$
2	0.683 479 5776	0.698 290 6029	0.849 105 6743	1.400 537 042	1.433 004 157	1.734 908 872
3	0.618 681 5919	0.636 117 3544	0.751 197 1947	1.635 310 944	1.688 309 710	2.002 783 856
4	0.586 053 7775	0.603 703 8516	0.697 632 4640	1.801 158 088	1.867 195 430	2.180 993 223
5	0.563 966 7759	0.581 052 2865	0.660 443 1817	1.930 546 670	2.005 181 489	2.313 446 064
6	0.546 784 8159	0.563 030 0215	0.631 624 1638	2.037 738 335	2.118 203 783	2.419 133 487
7	0.532 366 0850	0.547 686 1060	0.607 863 7335			

large. In table 7 we have given the predictions obtained for $m = \infty$ $(c = 1)$ and they are in very good agreement with the finite-size scaling estimates.

A comment about the finite-size scaling amplitudes $\bar{P}_2^{(3)}$ and $\bar{P}_3^{(3)}$ is in order. According to the Fradkin-Kadanoff (1980) ansatz (see equation (4.6)) one should expect the operator corresponding to $\bar{P}_2^{(3)}$ to have spin 1 and that corresponding to $\bar{P}_3^{(3)}$ to have spin $\frac{3}{2}$ but this is in contradiction with the estimates for $\bar{P}_2^{(3)}$ and $\bar{P}_3^{(3)}$ which are smaller than one and the trivial observation that $x > s$. As mentioned earlier (see § 4) a similar situation occurs in the four-state Potts model.

We now consider the eight-state model. In this case the normalised scaling amplitudes are not ε dependent and we have

$$\begin{aligned} &x_h = \bar{P}_0^{(1)} \simeq \bar{P}_0^{(2)} \simeq \bar{P}_0^{(4)} \qquad x_t = \bar{R}^{(0)} \\ &\tfrac{1}{2}x_{h,s} = \bar{P}_1^{(F)} \simeq \bar{P}_2^{(F)} \simeq \bar{P}_4^{(F)} \\ &x(\tfrac{1}{8}) = \bar{P}_1^{(1)} \qquad x(\tfrac{2}{8}) = \bar{P}_1^{(2)} \qquad x(\tfrac{3}{8}) = \bar{P}_1^{(3)} \\ &x(\tfrac{4}{8}) = \bar{P}_1^{(4)} \simeq \bar{P}_2^{(2)}. \end{aligned} \tag{5.9}$$

The estimates for the normalised scaling amplitudes are given in table 8. We again have to look for the central charge and use figures 1 and 3. We notice that for $m = 16$ the agreement between the estimates and the predictions obtained from conformal invariance is very good.

Table 4. Finite-size scaling amplitudes for the eight-state model with $\varepsilon = 0$.

(*a*)

N	$R^{(0)}$	$R^{(2)}$	$R^{(4)}$	$R^{(F)}$
2	3.061 467 459	2.472 135 955	4.000 000 000	2.947 251 516
3	2.854 395 782	3.251 227 892	4.207 699 618	3.411 417 226
4	2.677 198 642	3.625 200 209	4.553 971 545	3.663 167 754
5	2.533 203 176	3.842 612 950	4.697 270 784	3.821 500 222
6	2.413 732 253	3.985 107 864	4.768 375 005	3.931 166 251
7	2.312 171 250	4.086 433 952	4.807 938 105	4.012 479 109

(*b*)

N	$P_1^{(1)}$	$P_2^{(1)}$	$P_3^{(1)}$	$P_4^{(1)}$	$P_2^{(2)}$	$P_4^{(2)}$	$P_4^{(4)}$
2	1.858 333 539	2.094 730 847	2.211 168 973	3.226 251 859	2.397 824 735	3.226 251 859	5.226 251 859
3	1.905 891 474	2.192 453 335	2.290 637 024	3.289 118 590	2.600 803 603	3.225 770 537	4.953 280 227
4	1.959 697 294	2.263 233 176	2.342 839 300	3.305 877 888	2.720 699 517	3.216 285 202	4.720 492 063
5	2.009 363 385	2.318 979 639	2.384 777 842	3.313 049 180	2.802 758 926	3.208 126 517	4.547 587 650
6	2.054 405 377	2.365 486 018	2.421 211 886	3.318 240 162	2.864 349 588	3.202 247 456	4.415 973 950
7	2.095 491 914	2.405 792 977	2.454 073 075	3.323 393 588	2.913 510 850	3.198 439 782	4.312 780 773

(*c*)

N	$P_1^{(0)}$	$P_2^{(0)}$	$P_4^{(0)}$	$P_1^{(F)}$	$P_2^{(F)}$	$P_4^{(F)}$
2	0.732 292 6521	0.754 115 9045	1.226 251 859	1.368 346 484	1.443 131 285	2.271 558 410
3	0.673 371 1278	0.703 323 0103	1.039 241 511	1.578 371 581	1.694 367 711	2.484 828 378
4	0.643 603 4762	0.676 241 3556	0.942 348 3862	1.724 706 028	1.865 230 566	2.613 976 445
5	0.623 815 4796	0.656 970 2419	0.878 964 4422	1.837 824 848	1.993 917 262	2.704 458 501
6	0.608 768 6648	0.641 481 6515	0.832 471 0852	1.930 834 413	2.097 212 781	2.774 051 766
7	0.596 414 9722	0.628 241 5989	0.795 971 1875	2.010 447 330	2.183 775 554	2.831 010 485

Table 5. Estimates for the asymptotic (large N) values of the finite-size scaling amplitude: the six-state model.

ε	0	$\frac{1}{3}$	$\frac{3}{5}$
$R^{(0)}$	2.95-3.15	2.10-2.40	1.80-2.30
$R^{(2)}$	4.75-4.95	4.75-4.90	4.60-4.70
$R^{(3)}$	4.75-5.10	4.80-5.00	4.70-5.00
$R^{(F)}$	4.65-4.75	—	4.50-4.60
$P_0^{(1)})$	0.48-0.51	0.50-0.53	0.51-0.54
$P_0^{(2)}$	0.90-0.93	0.66-0.72	0.56-0.62
$P_0^{(3)}$	1.00-1.10	0.69-0.75	0.57-0.63
$P_1^{(1)}$	1.35-1.45	1.85-1.95	2.15-2.30
$P_1^{(2)}$	2.30-2.40	2.55-2.70	2.60-2.75
$P_1^{(3)}$	3.10-3.15	3.10-3.15	3.00-3.15
$P_2^{(2)}$	4.00-4.30	3.30-3.50	2.90-3.15
$P_2^{(3)}$	4.05-4.30	3.30-3.45	2.85-3.00
$P_3^{(3)}$	4.70-5.00	3.90-4.00	3.45-3.55
$P_1^{(F)}$	1.55 ± 1.70	—	2.20-2.40
$P_2^{(F)}$	2.75 ± 2.95	—	2.50-2.75
$P_3^{(F)}$	3.15 ± 3.35	—	2.65-2.90

Table 6. Estimates for the asymptotic (large N) values of the finite-size scaling amplitude: the eight-state model.

ε	0	$\frac{1}{3}$	$\frac{3}{5}$
$R^{(0)}$	1.60–2.00	1.35–1.70	1.20–1.55
$R^{(2)}$	4.65–4.80	4.55–4.70	4.50–4.65
$R^{(4)}$	4.70–4.85	4.60–4.75	4.55–4.70
$R^{(F)}$	4.45–4.60	—	4.30–4.45
$P_0^{(1)}$	0.52–0.57	0.47–0.52	0.45–0.50
$P_0^{(2)}$	0.54–0.60	0.48–0.53	0.45–0.50
$P_0^{(4)}$	0.56–0.60	0.49–0.54	0.45–0.50
$P_1^{(1)}$	2.25–2.50	2.35–2.60	2.42–2.67
$P_1^{(2)}$	2.52–2.77	2.60–2.85	2.62–2.87
$P_1^{(3)}$	2.52–2.77	2.60–2.85	2.65–2.90
$P_1^{(4)}$	3.33–3.40	3.30–3.40	3.25–3.35
$P_2^{(2)}$	3.15–3.30	3.10–3.30	3.10–3.30
$P_2^{(4)}$	3.15–3.25	2.95–3.10	2.85–3.05
$P_4^{(4)}$	3.50–3.85	3.30–3.50	3.05–3.25
$P_1^{(F)}$	2.55–2.90	—	2.75–3.10
$P_2^{(F)}$	2.80–3.10	—	2.90–3.20
$P_4^{(F)}$	3.20–3.40	—	3.10–3.40

Table 7. Estimates for the normalised finite-size scaling amplitudes for the six-state model. The predictions correspond to the scale dimensions obtained from the Virasoro algebra with $m=\infty$ ($c=1$).

	Estimates			Predictions
ε	0	$\frac{1}{3}$	$\frac{3}{5}$	$m=\infty$
$\bar{R}^{(0)}$	0.62 ± 0.06	0.47 ± 0.09	0.43 ± 0.15	0.5
$\bar{R}^{(2)}$	0.99 ± 0.05	1.00 ± 0.03	0.99 ± 0.03	1
$\bar{R}^{(3)}$	1.00 ± 0.05	1.00 ± 0.05	1.03 ± 0.09	1
$\bar{R}^{(F)}$	0.96 ± 0.05	—	0.97 ± 0.03	1
$\bar{P}_0^{(1)}$	0.102 ± 0.005	0.11 ± 0.01	0.11 ± 0.01	0.125
$\bar{P}_0^{(2)}$	0.19 ± 0.01	0.14 ± 0.02	0.12 ± 0.02	0.125
$\bar{P}_0^{(3)}$	0.21 ± 0.03	0.15 ± 0.02	0.13 ± 0.02	0.125
$\bar{P}_1^{(1)}$	0.29 ± 0.03	0.40 ± 0.03	0.48 ± 0.06	0.514
$\bar{P}_1^{(2)}$	0.5 ± 0.1	0.54 ± 0.05	0.57 ± 0.06	0.555
$\bar{P}_1^{(2)}$	0.63 ± 0.06	0.65 ± 0.05	0.65 ± 0.05	0.625
$\bar{P}_2^{(2)}$	0.85 ± 0.1	0.70 ± 0.05	0.65 ± 0.05	0.722
$\bar{P}_2^{(3)}$	0.85 ± 0.1	0.70 ± 0.05	0.60 ± 0.05	—
$\bar{P}_3^{(3)}$	1.0 ± 0.1	0.85 ± 0.05	0.75 ± 0.05	—
$\bar{P}_1^{(F)}$	0.35 ± 0.1	—	0.50 ± 0.05	0.50
$\bar{P}_2^{(F)}$	0.60 ± 0.1	—	0.55 ± 0.05	0.50
$\bar{P}_3^{(F)}$	0.65 ± 0.1	—	0.60 ± 0.05	0.50

We would like to stress that assigning $m=\infty$ to the six-state model and $m=16$ to the eight-state model is, using the language of experimental physicists, very preliminary. More precise estimates for the finite-size scaling amplitudes are necessary to fix m.

Table 8. Estimates for the normalised finite-size scaling amplitudes for the eight-state model. The predictions correspond to the scale dimensions obtained from the Virasoro algebra with $m = 16$.

	Estimates	Predictions $m = 16$
$\bar{R}^{(0)}$	0.33 ± 0.04	0.411
$\bar{R}^{(2)}$	0.985 ± 0.01	1
$\bar{R}^{(4)}$	1.00 ± 0.01	1
$\bar{R}^{(F)}$	0.95 ± 0.02	1
$\bar{P}_0^{(1)}$	0.11 ± 0.01	0.116
$\bar{P}_1^{(1)}$	0.52 ± 0.04	0.536
$\bar{P}_1^{(2)}$	0.57 ± 0.05	0.558
$\bar{P}_1^{(3)}$	0.57 ± 0.05	0.595
$P_1^{(4)}$	0.71 ± 0.05	0.647
$P_2^{(2)}$	0.68 ± 0.03	0.647
$P_2^{(4)}$	0.65 ± 0.02	—
$P_4^{(4)}$	0.72 ± 0.04	—
$\bar{P}_1^{(F)}$	0.65 ± 0.10	0.562

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